

ON THE PERFORMANCE AND IMPLEMENTATION OF A CLASS OF ERROR AND ERASURE CONTROL (d,k) BLOCK CODES

C.S. COETZEE, H.C. FERREIRA AND P.G.W. VAN ROOYEN

Cybernetics Laboratory, Rand Afrikaans University, P.O. Box 524, Johannesburg, 2000, South Africa

Abstract — We investigate the error performance and implementation of a class of error and erasure correcting (d,k) block codes. These codes can be represented by restricted integer compositions of an arbitrary length, appended with a small and fixed number of redundant bits. Bounds on block error rates, as well as error probability curves obtained by simulations are presented.

INTRODUCTION

Digital recording systems usually employ a concatenation of an outer error control code and an inner (d,k) modulation code. Recently, "combined" codes capable of error control as well as meeting (d,k) input restrictions on the storage medium have been investigated [1-8]. For the trellis coding schemes [1-3] this usually resulted in code rates significantly lower than the overall rates of concatenated schemes.

In this paper the performance of error and erasure correcting (d,k) block codes capable of achieving high rates are investigated. Codes designed for correcting single errors and up to $d + 1$ adjacent erasures are considered, as well as codes correcting double adjacent errors and codes correcting up to $2d + 2$ adjacent erasures. Block error rates resulting from computer simulations of these codes on various channel models are presented, and compared to bounds in certain cases. In the case of the (d,k) = (1,2) single error correcting code, a comparison was also made between results obtained by simulation of both the theoretical decoding algorithm and a hardware implementation.

The codes investigated here were synthesized after applying the principles of integer partitioning. Error correcting properties result when certain compositional restrictions are imposed on (d,k) sequences and when parity bits separated by buffer bits are appended [6,7].

CODE CONSTRUCTIONS

With each (d,k) sequence of length ℓ bits, a composition, $\Phi = \phi_1 \phi_2 \dots \phi_r$, of an integer ℓ , ie an ordered collection of r smaller integers, or parts ranging between $d + 1$ and $k + 1$ and summing to ℓ , can be associated. Each "1" in the (d,k) sequence denotes the start of a new part. Henceforth we shall refer to such (d,k) sequences onto which information bits are directly mapped at the encoder, as composition blocks.

For error correction, all composition blocks are restricted so as not to allow the simultaneous inclusion of two consecutive parts a and b together with their arithmetic sum $c = a + b$. The following set of parts satisfy this basic composition rule.

$$X = \{x_i | d + 1 = x_1 \leq x_i \leq x_w = k + 1, k = 2d\} \quad (1)$$

Single errors will always be detected within the composition block, since a violation of either the d or k constraint is always implied. Furthermore, such an error will always result in one or more parts ϕ_i , $\phi_i \notin X$, and will either reduce or increase r by one unit. Single error correction can then be achieved with d parity bits, computed from:

$$p_j = \sum_i i \text{ modulo } 2 \quad \forall i \ni \phi_i = x_j \quad (2)$$

where $1 \leq j \leq v = d$, $1 \leq i \leq r$. Finally, a codeword may be represented by the following binary string:

$$s_1 s_2 \dots s_r p_1 b_1^1 b_1^2 \dots b_1^d p_2 b_2^1 \dots b_2^d \dots p_v b_v^1 \dots b_v^d$$

The composition block corresponds to the bits s_i , $1 \leq i \leq \ell$; p_i represents the parity bits and the b_i^j , $1 \leq j \leq d$, are buffer bits. The buffer bits are initially set to "0" to ensure compliance with the d constraint between parity bits and may later be inverted to "1" to reduce the overall k constraint of the codeword. The total codeword length, n , is thus given by

$$n = \ell + d^2 + d. \quad (3)$$

The parity bits are recomputed at the receiver and compared with those sent, aiding in the correction of some errors.

To correct double adjacent errors, the set of parts in (1) is limited by deleting every second part in X , starting with x_2 . Parity bits are again computed as in (2).

For erasure correction, k need not be restricted, ie $k > 2d$. Up to $d + 1$ adjacent erasures can be corrected with the aid of the parity bits in (2). Correct decoding of up to $2d + 2$ adjacent erasures is possible, using $3d + 3$ parity bits computed as follows:

$$p_j^o = r_j \text{ modulo } 2 \quad (4)$$

$$p_j^1 p_j^2 = \sum_i i \text{ modulo } 4 \quad \forall i \ni \phi_i = x_j \quad (5)$$

where $1 \leq j \leq w = d + 1$, $1 \leq i \leq r$, and r_j is the number of occurrences of x_j in the composition block. The length of a codeword then becomes

$$n = \ell + 3d^2 + 6d + 1. \quad (6)$$

Since d is usually small, it follows from (3) and (6) that the code rates can approach the capacity of the (d,2d) constrained channel for large ℓ .

IMPLEMENTATION OF ENCODER AND DECODER

At the encoder, a simple (d,k) finite state machine maps information bits onto composition blocks. The parity bits are then computed and appended to the composition block. Suitable buffer bits are inserted between the parity bits. The received composition block is restored at the decoder such that (d,k) integrity is achieved and such that the recomputed parity bits agree with those received. In certain cases, unambiguous errors (ie drop in of "1" into $\phi_i = x_1$ or drop out of "1" between $\dots \phi_i \phi_{i+1} \dots = \dots x_w x_w \dots$) can be corrected without recourse to the parity bits, merely by ensuring (d,k) compliance. The block diagram of a hardware encoder and decoder is shown in Fig. 1 and Fig. 2 respectively. Note that x_w is the minimum length of the shift register in Fig. 1.

CHANNEL MODELS

The codes were firstly investigated on the binary symmetric and binary symmetric erasure channels (BSC and BSEC). A Gilbert burst channel is shown in Fig. 3a. In the good state, the output of the encoder is presented without change to the decoder. The probability of error in the bad state of the hard decision burst channel is 0.5. For the burst erasure channel in the bad state, both a "1" and a "0" at the encoder output is received as an erasure by the decoder. A ternary bisymbol sliding window channel is shown in Fig. 3b. For the random peak shift channel (RPSC), $p = q$. A unidirectional shift channel implies either $p = 0$ or $q = 0$.

BOUNDS ON PERFORMANCE

Bounds on block error rates can readily be established. For single error correction, it is assumed that two or more errors introduced in the composition block and parity bits will result in incorrect decoding of the block, except if all those errors are present only in the parity bits. Errors in the buffer bits do not affect decoding. Also, one unambiguous error in the composition block, with the other errors located in the parity bits only, would not influence decoding. Disregarding the latter case, an upper bound on the BSC would therefore be

$$\bar{P}_{\text{block}} = \sum_{i=2}^{\ell+d} \left[\binom{\ell+d}{i} - \binom{d}{i} \right] p^i (1-p)^{\ell+d-i} \quad (7)$$

For erasure correction on the BSEC and double adjacent error correction on the BSC we introduce a variable h , equal to either $d+1$, $2d+2$ or 2 as determined by the correcting capability of the code. Then, on the BSEC, we have

$$\bar{P}_{\text{block}} = \sum_{i=2}^{\ell+v} \left[\binom{\ell+v}{i} - \binom{v}{i} - K_i(\ell-i+1) \right] p^i (1-p)^{\ell+v-i} \quad (8)$$

where $K_i = 1$ for $i \leq h$ and $K_i = 0$ otherwise.

Double adjacent error correction is most effective on the random peak shift channel, where we propose both an upper and a lower bound:

$$P(\text{bound}) = \sum_{i=2}^{\delta} \binom{\delta}{i} p_i (1-p)^{\delta-i} \quad (9)$$

For the upper bound $\delta = \lfloor \ell/(d+1) \rfloor + v$, the maximum number of possible peak shifts in a codeword, and similarly $\delta = \lfloor \ell/k + 1 \rfloor$ for the lower bound.

Bounds for the Gilbert channel tend to be more complex and are not derived.

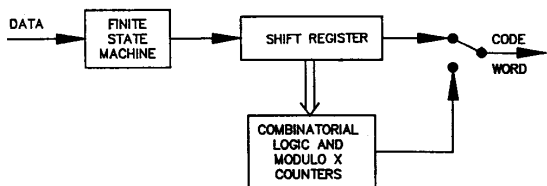


Figure 1. General Encoder

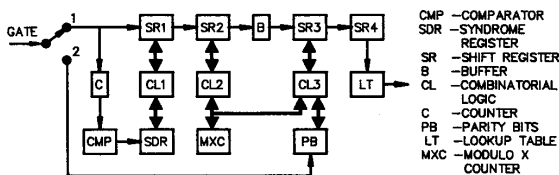


Figure 2. General Decoder.

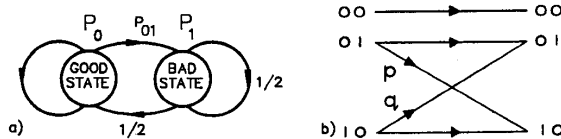


Figure 3. a) Gilbert Burst Channel. b) Random Peak Shift Channel.

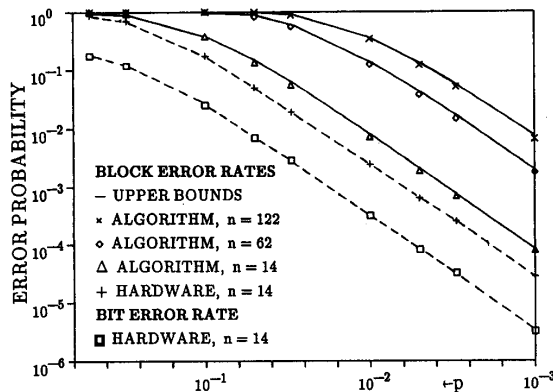


Figure 4. Single Error Correction on Binary Symmetric Channel $(d,k) = (1,2)$

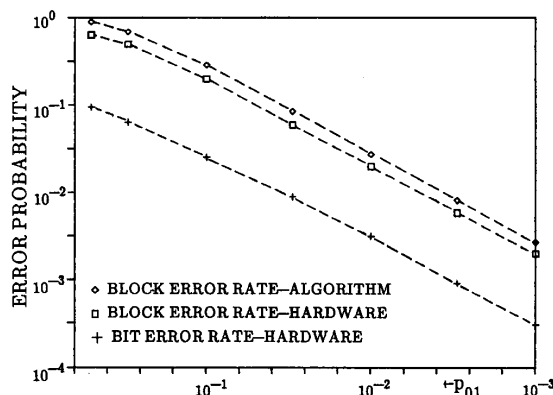


Figure 5. Single Error Correction on Hard Decided Burst Channel $(d,k) = (1,2); n = 14$

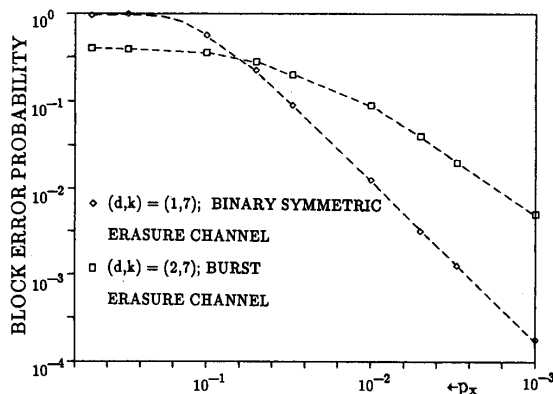


Figure 6. $d + 1$ Adjacent Erasure Correction, $\ell = 120$
 $P_x = p$ on BINARY SYMMETRIC ERASURE CHANNEL
 $P_x = p_{01}$ on BURST ERASURE CHANNEL

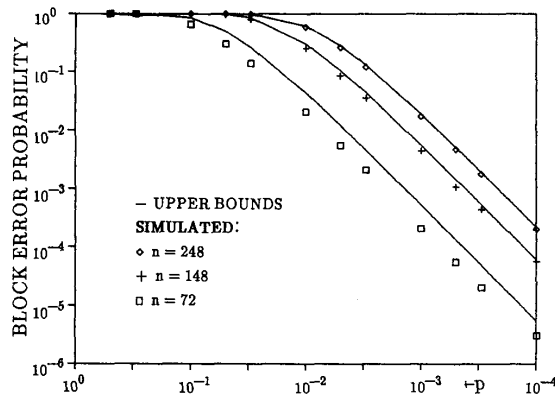


Figure 7. $2d + 2$ Adjacent Erasure Correction on Binary Symmetric Erasure Channel, $(d,k) = (3,6)$

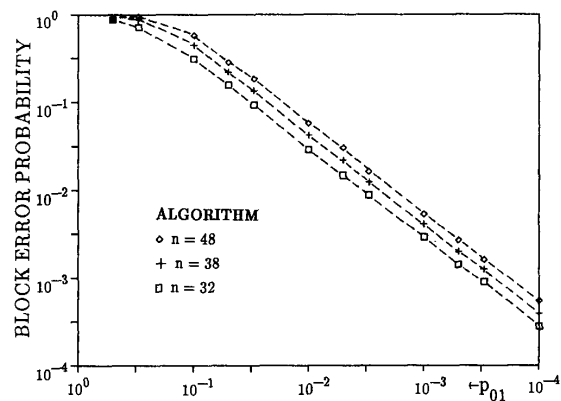


Figure 8. $d + 1$ Adjacent Erasure Correction on Burst Erasure Channel, $(d,k) = (4,8)$

d	ΔC (%)	ΔDR_1 (%)	ΔDR_2 (%)
4	15.41	19.9	9.6
5	12.76	11.3	0
6	10.91	24.0	14.3
7	9.53	33.5	25
8	8.47	40.9	33.3
9	7.62	33.5	25

$$\begin{aligned}\Delta C &= C(d,\infty) - C(d,2d) \\ \Delta DR_1 &= DR(d,2d) - DR(1,7) \\ \Delta DR_2 &= DR(d,2d) - DR(2,7)\end{aligned}$$

Table 1. Comparison of certain capacities and increases in density ratios for $(d,2d)$ codes; $d \geq 4$

RESULTS

The performance of the various code constructions with different values of d,k and ℓ are presented in Fig. 4-9. These simulation results also confirmed the bounds (7) to (9).

The hardware decoder for the $(d,k) = (1,2)$ code performed better than the theoretical algorithm (Fig. 4,5). This happened because of the sequential nature in which the hardware decoding stages were implemented. For instance, after correcting an unambiguous error, the theoretical decoder would not check for any further errors in the composition block. The hardware decoder, however, would still try to correct subsequent errors, in the same composition block, with the aid of parity bits.

The results of double adjacent error correction obtained on the random peak shift channel (Fig. 9) were identical to error rates obtained on unidirectional shift channels.

Contrary to expectation, use of the $(d,2d)$ constraint does not always result in a significant capacity loss, as can be seen from ΔC in Table 1. The density ratio, DR, is the minimum distance separation between transitions in the channel waveform, is equal to $(d+1)m/n$ data bit intervals, and poses a restriction on the maximum recording density. It is clear from Table 1 that as d increases, the DR of $(d,2d)$ codes exceeds the DR of the state of the art codes for which d is small and $k > 2d$, ie the $(d,k) = (1,7)$ and $(d,k) = (2,7)$ codes. Codes with a $(d,2d)$ constraint would thus be suitable for future systems with larger d constraints.

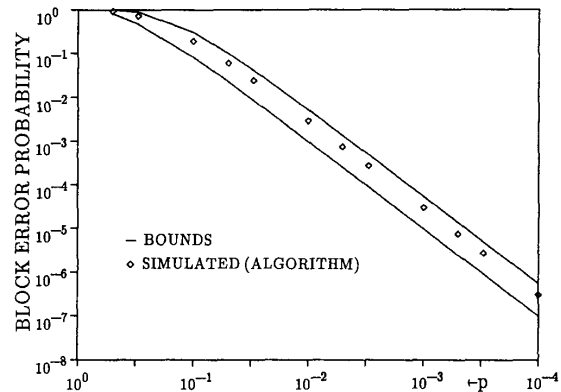


Figure 9. Double Adjacent Error Correction on Peak Shift Channel, $p = q$, $(d,k) = (8,16)$; $n = 106$

For erasure correction we do not need to impose the $k = 2d$ restriction on our constructions. Therefore, for the $(d,k) = (1,7)$ and $(d,k) = 2,7$ constraints (Fig. 6) we find that the code rates can approach the capacity of the d constrained channel for small d .

CONCLUSIONS

We have investigated a class of (d,k) block codes capable of error and erasure control. The codes were synthesized by imposing compositional restraints on (d,k) sequences and/or by adding a small number of parity and buffer bits to a codeword.

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