MAXIMUM ENTROPY INVESTIGATION OF THE INTER USER INTERFERENCE DISTRIBUTION IN A DS/SSMA SYSTEM

Pieter van Rooyen * and Fritz Solms ‡
* Electrical and Electronical Engineering Department, University of Pretoria, Pretoria, 0002, South Africa
‡ Department of Applied Mathematics, Rand Afrikaans University, Johannesburg, South Africa

ABSTRACT

The Maximum Entropy (MaxEnt) method is used to evaluate the Inter User Interference (IUI) probability density function in a Direct Sequence Spread Spectrum Multiple Access (DS/SSMA) system. This distribution is frequently assumed to be Gaussian distributed and is commonly known as the Gaussian Assumption (GA). By calculating the discrimination information (relative entropy) between the IUI-distribution, as inferred via the MaxEnt method, and a Gaussian distribution with equal second moments the Gaussian Assumption is quantified for the Nakagami-m faded channel. By altering the parameter m of the Nakagami-m distribution, the degree of fading can be varied and therefore the influence of fading on the Gaussian Assumption can be thoroughly investigated.

1. INTRODUCTION

To accurately calculate error probabilities of a DS/SSMA system is quite a momentous task and therefore some approximate models are often used to simplify computations. As mentioned, one of the most common approximations is to consider only second moment information of the IUI random variable. Unfortunately, large errors can result from casual and gratuitous appeal to this approximation, especially when the transmission channel is faded.

Most papers on the applicability and accuracy of the Gaussian Assumption only considered the unfaded case [1, 2]. Holtzman [3] gives a brief synopses of some recent results. In this paper the Gaussian Assumption is investigated under more general, fading conditions, using Nakagami-m channel fading statistics.

The Nakagami-m distribution [4] (for brevity called the Nakagami distribution hence forth) is a viable fading model which fits certain urban propagation data better than Rayleigh, Rician or log-normal distributions [5, 6], is a convenient model for characterizing the severeness of multipath fading. The parameter $1/2 \leq m \leq \infty$ and indicates severest to least fading respectively. This model assumes that the received signal is a sum of vectors with random strengths and random phases, which offers more flexibility than Rayleigh or Rician distributions. In fact, the Rayleigh distribution is a special case of the Nakagami distribution where the Nakagami parameter $m = 1$. As m increases the distribution tends to an impulse, constituting no fading. It is therefore meaningful to investigate the Gaussian Assumption under Nakagami fading conditions, due to its realistic fading statistics and flexibility in adjusting the amount of fading.

In a previous paper [7] it was shown that MaxEnt can be used to accurately infer a random variable's probability distribution from its moments. Therefore, by calculating the moments of the IUI random variable under Nakagami fading conditions, it is possible to quantify the accuracy of the Gaussian Assumption as a function of m.

In the next section an appropriate system model is proposed to investigate the Gaussian Assumption under fading conditions, Sections 3 and 4 respectively summarise the MaxEnt formulism and presents numerical results, quantifying the accuracy of the Gaussian Assumption. The paper is concluded in Section 5 by summarising some of the main results.

2. SYSTEM MODEL

A spread spectrum multipath system model for K users is indicated in Figure 1. The channel for the desired transmitter and receiver ($k = 1$) can be represented by an L- paths Nakagami fading model where a single transmitted pulse is received via L-paths at the random instant $t_l, l = 1, \cdots, L$. We assume t_l is uniformly distributed over one bit period $(0, T]$ and that each user code sequence has a period of $N = T/T_c$.

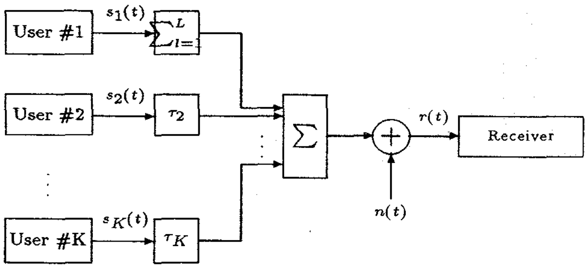

Figure 1: Transmission model

In the analysis we assume average power control, whic also includes averaging the channel fading characteristics. Baseband signalling at a rate less than the chann coherence bandwidth ensures that intersymbol interference can be neglected in the inferred IUI distribution. Therefore, the channel has a low-pass equivalent impulse response given by

$$h(t) = \sum_{l=1}^{L} \beta_l \delta(t - t_l) e^{j\phi_l}, \qquad (1)$$

where $\delta(\cdot)$ is the delta function, β_l is the Nakagami distributed path gain and ϕ_l is the random path phase, uniformly distributed between $(0, 2\pi]$.

In the transmission model it is further assumed that the kth interfering user of the multiple access system is linked to the receiver of Figure 1 via a single Nakagami fading path with a uniformly distributed random delay τ_k ranging from zero to one bit period, T. This will naturally result in a worst case scenario, rendering the results conservative.

In our formulation we specify the Nakagami distributed path gain of the $K-1$ interfering users by $V_k, k = 2, \cdots, K$. Thus, the received signal for the fading model described is given by

$$\begin{aligned} r(t) &= A \sum_{l=1}^{L} \beta_l a_1(t - t_l) b_1(t - t_l) \cos(\omega_c t + \Phi_l) \quad (2) \\ &+ A \sum_{k=2}^{K} V_k a_k(t - \tau_k) b_k(t - \tau_k) \cos(\omega_c t + \Psi_k) \\ &+ n(t) \end{aligned}$$

where $\Phi_l = \theta_1 - \omega_c t_l + \phi_l$, $\Psi_k = \theta_k - \omega_c \tau_k$ and θ_k the phase of the kth user. Also, $n(t)$ is white Gaussian noise with double sided spectral density of level $N_0/2$ and θ_1 can be assumed zero with no loss of generality.

Using this system model it is possible to derive moments necessary to infer the IUI probability density function. The moments used here are derived in [8].

3. MaxEnt Formulation

Shore and Johnson [9] have proven that MaxEnt [10] is the only method for inferring from incomplete information that does not lead to logical inconsistencies.

In the MaxEnt formulation the missing information (the information entropy) [11]

$$I(x) = -\int_a^b p(x) \ln p(x) dx \qquad (3)$$

is maximized subject to the constraints of the normalization of the pdf and subject to the available information. In our case the moments must be equal to the measured or calculated moments. This is a standard maximum entropy moment problem (the MaxEnt moment problem has been studied in detail by Tagliani [12]). The constraints are introduced via Lagrange multipliers and the resulting expression for the inferred pdf is

$$p(x) = \frac{1}{Z} \exp\left\{ -\sum_{m=1}^{M} \lambda_m x^m \right\} \qquad (4)$$

where the information about the normalization is contained in the partition function

$$Z = \int_a^b \exp\left\{ -\sum_{m=1}^{M} \lambda_m x^m \right\} dx \qquad (5)$$

and the Lagrange multipliers are determined by requiring that

$$\langle x^m \rangle = \mu_m = \int x^m p(x) dx \quad ; \quad m = 1, \ldots, M. \qquad (6)$$

Agmon et al [13] have noted that defining

$$F(\{\lambda_m\}) = \ln Z + \sum_{m=1}^{M} \lambda_m \mu_m \qquad (7)$$

yields

$$\frac{\partial F}{\partial \lambda_m} = \mu_m - \langle x^m \rangle. \qquad (8)$$

Hence minimizing F is equivalent to solving the set of coupled nonlinear equations of (6). Furthermore, the authors [13] have shown that the Hessian matrix $\mathbf{H}$ with

$$H_{mm'} = \frac{\partial^2 F}{\partial \lambda_m \partial \lambda_{m'}} = \langle x^{m+m'} \rangle - \langle x^m \rangle \langle x^{m'} \rangle \qquad (9)$$

is positive definite and thus that F is a strictly convex function of the Lagrange multipliers $\{\lambda_m\}$. Consequently F has a unique minimum (i.e. (6) has a unique solution) and a Newton-Raphson minimization procedure [14] is guaranteed to converge. Define an error vector

$$\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_M)^T \quad : \quad \epsilon_m \equiv \mu_m - \langle x^m \rangle \qquad (10)$$

and let $\bar{\lambda} = (\lambda_1, \ldots, \lambda_M)$. Then the new guess after a Newton Raphson step is

$$\bar{\lambda}' = \bar{\lambda} - \mathbf{H}^{-1} \cdot \bar{\epsilon}. \qquad (11)$$

During each iteration a set of coupled linear equations is solved for the Newton step $\overline{\delta} = \overline{\lambda} - \overline{\lambda}'$

$$\mathbf{H} \cdot \overline{\delta} = \overline{\epsilon}. \tag{12}$$

Since $\mathbf{H}$ is positive definite it is also non-singular. Equation (12) is solved with a standard LU-decomposition with a back-substitution algorithm coded in C^{++}.

Also included in the code to combat spurious divergence due to computer round-off errors, the "globally convergent" Newton- Raphson method found in *Numerical Recipes in C* [15] was translated into C^{++} using vector and matrix classes, resulting in an increase of the number of Lagrange multipliers to about 24.

4. Numerical Results

In has been shown that the MaxEnt approach produces credible estimates of the distribution of a random variable under a variety of conditions [7]. It is hence possible to reliably infer the IUI distribution and to quantify the validity of the GA by calculating $\mathcal{I}$, the relative missing information. Also, the influence of the Nakagami fading parameter m on the validity of the GA is investigated. (As explained earlier $m = \frac{1}{2}$ corresponds to the case where the signal suffers severest fading, while the condition $m = \infty$ indicates no fading.)

For the unfaded case ($m = \infty$), Table 1 signifies the moments, for $K = \{2, 25\}$ and $N = 127$. Only the even moments are indicated since the IUI is a symmetric, zero mean random variable [1, 16] and hence all odd moments are zero. These moments were generated with Gold spreading sequences. Also tabulated are the moments of a Gaussian distribution with variance equal to the second moment of the IUI variable. It is clear that the moments for this condition is closer to a Gaussian distribution than for $K = 2$.

Although the pdf of the IUI is not of much significance other than to calculate $\mathcal{I}$ and the average probability of error, let us nevertheless examine the distribution for $K = \{2, 25\}$, $N = 127$ and $m = \infty$ as indicated in Figures 2 and 3 respectively. These distributions were generated using 10 moments of the IUI random variable.

For $K = 2$ (Figure 2) it is apparent that the distribution of the IUI is not very close to a Gaussian distribution. The IUI distribution has a lot more structure and does not follow the Gaussian tail at all. However, as intuitively expected from the central limit theorem, the IUI random variable approaches the Gaussian distribution much better as K increases - as indicated in Figure 3 for $K = 25$ only the tail distribution of the IUI random variable does not follow the Gaussian distribution exactly.

By looking at the moments, as in Table 1, or the actual distribution, as in Figures 2 and 3, of the IUI random variable, it is apparent that as K increases the GA becomes more valid. However, these measures do not give a generalised or quantative indication of the validity of the GA. By calculating $\mathcal{I}$, the relative entropy, we have a dimensionless quantity that fulfil the criteria of a generalized parameter with $0 \leq \mathcal{I} \leq \frac{1}{2}$. If $\mathcal{I} = 0$ the two random variables are 100% identically distributed. On the other hand, when $\mathcal{I} = \frac{1}{2}$ the missing information is a maximum.

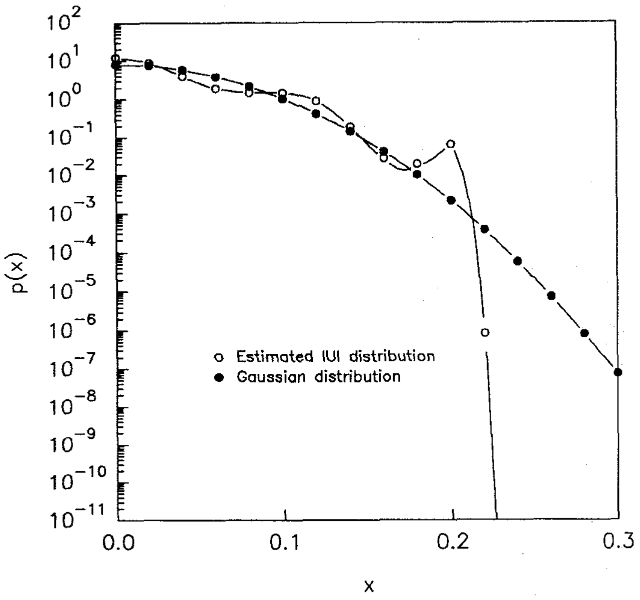

Figure 2: IUI distribution for $K = 2, N = 127, N_m = 10$ and $m = \infty$

Figure 4 shows $\mathcal{I}$ between the "exact" IUI-pdf and a Gaussian distribution with the same variance σ^2_{ma} for $m = \{\frac{1}{2}, 5, \infty\}$. Irrespective of m, the missing information between the two distributions decreases exponentially as K increases. As m decreases, in other words as the channel becomes more faded, the IUI distribution is less Gaussian.

Further, for the Gold sequences adopted in this work, the maximum family size is 65 for $N = 127$. From Figure 4 it is clear that the missing information, $\mathcal{I}$, saturates at roughly 10^{-5} for $m = \infty$ and 10^{-3} for $m = \frac{1}{2}$. This observation is quite significant since an increase in the number of users, that is an increase in K, does not imply a more Gaussian nature of the IUI random variable. The saturation of $\mathcal{I}$ suggest that the GA is only valid at high signal-to-noise ratios or high average error rates, especially under faded conditions.

Table 2 gives an indication of the influence of the chipping sequence length N for $m = \{\frac{1}{2}, 5, \infty\}$. To investigate the influence of N the ratio $\frac{K}{N}$ has to be constant for different values of N. As N increases, $\mathcal{I}$ decreases, indicating that the IUI random variable becomes more Gaussian as N increases. As for an increase in K, $\mathcal{I}$ also saturates at a specific value, indicating that not even an

N_m	$N, = 127$ $K = 2$	Gaussian moments	$N = 127$ $K = 25$	Gaussian moments
2	0.00242647	0.00242647	0.063177	0.063177
4	2.45809e-05	1.76633e-05	0.0121269	0.011974
6	4.08285e-07	2.14297e-07	0.00392591	0.00378241
8	9.56568e-09	3.6399e-09	0.00179922	0.00167273
10	2.86238e-10	7.94889e-11	0.00107131	0.000951101
12	1.00233e-11	2.12165e-12	0.000787382	0.000660965
14	3.86378e-13	6.69256e-14	0.000690347	0.000542851
16	1.58152e-14	2.43589e-15	0.000704616	0.000514436
18	6.73614e-16	1.00481e-16	0.000821985	0.000552509
20	8.34723e- 18	4.63245e-18	0.000872412	0.000663211

Table 1: Moments of the IUI random variable for $K = \{2, 25\}$, $N = 127$ and $m = \infty$

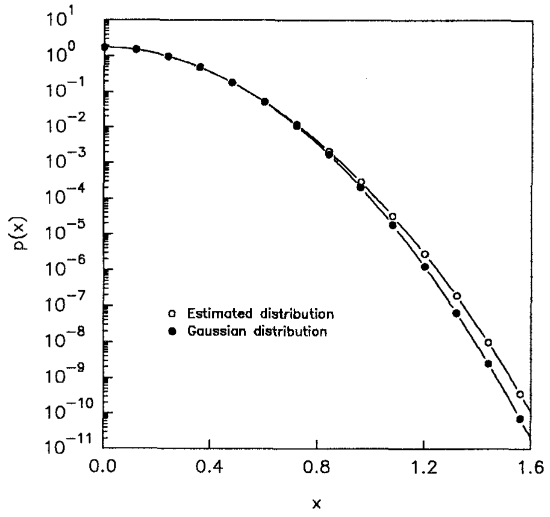

Figure 3: IUI distribution for $K = 25, N = 127$, $N_m = 10$ and $m = \infty$

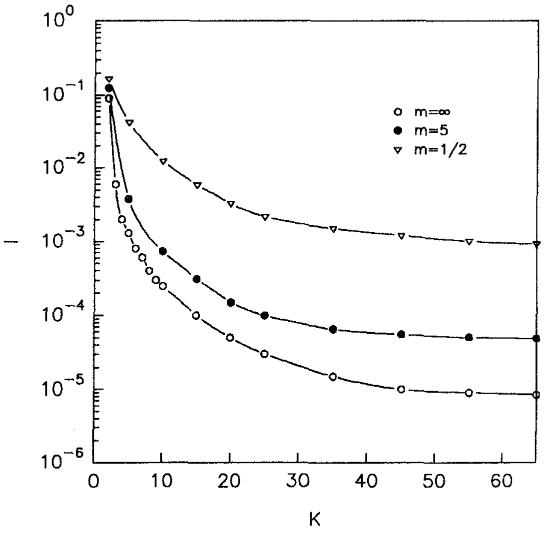

Figure 4: Relative Entropy, $\mathcal{I}$, of the IUI distribution, $m = \{\frac{1}{2}, 5, \infty\}$

infinite increase in N will result in a perfect Gaussian distribution. When fading is added to the signal, that is low values of m, the distribution is even less Gaussian.

It can be concluded that for the Gold spreading sequences used in this study, the GA is inaccurate, especially when a fading channel is considered.

Having an accurate description of the IUI probability distribution function, the average probability of error of a spread spectrum multiple access system can be calculated. Figure 5 indicates the average error probability when calculated with the MaxEnt principle. Also indicated in the figure is a comparison with other techniques, such as Gauss Quadrature Rules (GQR), making use of the Gaussian Assumption and simulations, to calculate the average error rate. Interesting to note is that MaxEnt is numerical more stable than the GQR technique at high SNR. This is due to the well known numerical stability problems of the GQR technique.

5. CONCLUSION

In this paper the accuracy and validity of the Gaussian Assumption were investigated under Nakagami fading conditions using the Maximum Entropy principle. A system model was further proposed to calculate moments.

The missing information between the IUI distribution and a Gaussian distribution with the same second moment were calculated and presented in a number of tables and graphs. The main conclusion to be drawn from this study is that the missing information $\mathcal{I}$ for $m = 1/2$ (severest fading) is almost three orders of magnitude more than for the unfaded case, $m = \infty$. Therefore, the Gaussian Assumption is significantly more valid under unfaded condition and more exact methods than the Gaussian Assumption has to be used to calculate error probabilities in faded DS/SSMA systems - methods such as MaxEnt can be implemented very effectively in calculating accurate average error rates.

		$\mathcal{I}$		
K	N	$m = 1/2$	$m = 5$	$m = \infty$
2	31	0.12616	0.11971	0.01190
3	63	0.03171	0.02157	0.01154
5	127	0.01929	0.00613	0.00141
9	255	0.01348	0.00281	0.00049
17	511	0.00974	0.00128	0.00021

Table 2: Relative entropy, $\mathcal{I}$, for $m = \{\frac{1}{2}, 5, \infty\}$ and $\frac{K}{N}$ constant

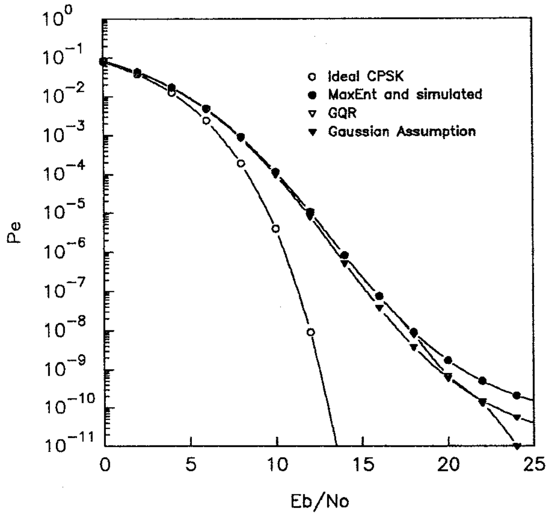

Figure 5: MaxEnt compared with GQR and computer simulations for CPSK; $m = \infty$, $K = 10$ and $N = 127$

References

[1] M. Pursley, "Performance Evaluation for Phase-Coded SSMA Communications - Part 1: System Analysis," *IEEE Transactions on Communications*, vol. COM-25, no. 8, pp. 795–803, August 1977.

[2] J. Lehnert and M. Pursley, "Error probabilities for binary ds/ss communications with random signature sequences," *IEEE Transactions on Communications*, vol. COM-35, pp. 87–98, January 1987.

[3] J. K. Holmes, *Coherent Spread Spectrum Systems.* Wiley and Sons, New York, 1982.

[4] P. Crepeau, "Uncoded and coded performance of MFSK and DPSK in Nakagami fading channels," *IEEE Transactions on Communications*, vol. 40, pp. 487–493, March 1992.

[5] H. Xiang, "Binary Code-Division Multiple-Access Systems operating in Multiple Fading, Noisy channels," *IEEE Transactions on Communications*, vol. COM-33, no. 8, pp. 775–784, August 1985.

[6] M. A. Mokhtar and C. Gupta, "Power control considerations for DS/CDMA personal communication systems," *IEEE Transactions on Vehicular Technology*, vol. 41, pp. 479–487, November 1992.

[7] F. Solms, P. G. W. van Rooyen, and J. Kunicki, "Maximum entropy and minimum relative entropy in performance evaluation of digital communication systems," *IEE Proceedings-I.* accepted for publication.

[8] P. G. W. van Rooyen, *Maximum Entropy based analysis of a DS/SSMA diversity system.* PhD thesis, Rand Afrikaans University, Johannesburg, November 1994.

[9] J. E. Shore and R. W. Johnson, "Axiomatic derivation of the principle of Maximum Entropy and the principle of Minimum Cross-Entropy," *IEEE Transactions on Information Theory*, vol. IT-26, no. 26, pp. 26–37, 1980.

[10] E. T. Jaynes, "Information theory and statistical mechanics-I," *Physics Review*, vol. 106, pp. 620–630, 1957.

[11] C. E. Shannon, "A mathematical theory of communications," *Bell systems technical journal*, no. 27, p. 379, 1948.

[12] A. Tagliani, "On the application of Maximum Entropy to the moments problem," *Journal on Mathematical Physics*, no. 34, pp. 326–337, 1993.

[13] N. Agmon, Y. Alhassid, and R. Levine *Journal on Computational Physics*, no. 30, p. 250, 1979.

[14] L. Mead and N. Papanicolaou, "Maximum Entropy in the Problem of Moments," *Journal on Mathematical Physics*, vol. 25, pp. 2404–2417, August 1984.

[15] W. Press, W. Flannery, and S. Teukolski, *Numerical recipes in C.* Cambridge, UK: Cambridge university press, 2nd ed., 1992.

[16] K. Morletta and L. Lennert, "On the calculation of average error rates in spread spectrum systems," *Proceedings of ISSSTA '92*, pp. 25–29, 1992.